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Quantum relative states

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Abstract. We study quantum state estimation problems where the reference system with respect to which the state is measured should itself be treated quantum mechanically. In this situation, the difference between the system and the reference tends to fade. We investigate how the overlap between two pure quantum states can be optimally estimated, in several scenarios, and we re-visit homodyne detection.

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1 Introduction

Today, entanglement is widely recognized as one of the characteristic feature, if not the essence, of quantum physics [1,2]. Historically, the experimental paradigm of entanglement has long been Bell tests [3,4], that is, experiments that demonstrate quantum non-locality. A more modern example is quantum teleportation [5]. The beauty of teleportation is that a quantum state dissolves at some point of space-time and appears at another without ever existing in-between. Indeed, neither the quantum system held by the receiver, nor the classical information, that makes teleportation possible, contain any information about the original state.

A fundamental problem tackled by quantum information Science is to characterize entangled *states*, and much has been learnt along this line of research [2]. However, entanglement is also a feature of some measurements, that we will refer to as coherent measurements, characterized by self adjoint operators whose eigenvectors are entangled states. Interestingly, both aspects of entanglement, i.e. states exhibiting quantum correlations and coherent measurements, are exploited in an essential way in quantum teleportation[1].

A property of coherent measurements, that we will be interested in here, is that they allow to measure *relative* properties of a set of quantum systems without gaining information about the individual subsystems. In contrast, there is no non-trivial manner to measure a relative property of classical systems without actually measuring each system and computing the relative property from the measurement outcomes. For example, given two classical arrows, there is no way to find out the angle between the arrows without gaining information about the direction of each arrow (at least in principle, in practice one can always forget about classical information). What happens, in this classical setting, is that the first direction is macroscopic enough that it serves as a reference axis, with respect to which the direction of the second arrow is measured. This measurement can be performed with very high precision since the second arrow is macroscopic enough that it can be considered classical.

Every measurement on a quantum system can be thought of as a measurement of some property of this system with respect to a reference system. This reference system is usually that macroscopic that one can safely disregard its quantum properties. We here want to consider

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[1] Despite all the limitations of intuition in quantum physics, let us elaborate on this. Roughly, entanglement provides the parties with correlations strong enough for them to be able to always give the same answer whenever asked the same question (the typical singlet state always provides opposite measurement results, whenever measurement along parallel axes are performed). Now, the coherent Bell measurement, used in quantum teleportation, does something like asking "*how similar they you?*" to one of the entangled system and to the system to be teleported. If the answer happens to be: *we are alike*, i.e. would we receive the same question, we would give the same answer, then, clearly, Bob's system would response to any measurement in the same way the original system would, hence achieving teleportation. Next, if the answer is: we are alike up to a standard symmetry, then teleportation succeeds as soon as the receiver, Bob, gets the information about which standard symmetry. This illustrates how quantum teleportation exploits the dual aspect of entanglement, i.e. that both aspects of entanglement are equally essential although one aspect received much more attention than the other, because of the historical role played by Bell inequalities.

the case where *both* the reference system and the measured system are treated quantum mechanically. In a sense, we want to "quantise" the reference system of a quantum measurement. Note that a lot of attention has been devoted recently to similar problems. We could cite for example the issue of encoding a direction or a Cartesian frame into a quantum object [6], or the design of programmable universal measurement devices [7].

We will treat two classes of relative state estimation problems: the quantum analogue of the estimation of the angle between two arrows, and homodyne measurements. In Section 2, we will address the issue of optimally estimate the (modulus of the) scalar product between two qubit states, extending on the work of reference [8]. We will consider (i) the situation where each qubit state is represented by identically prepared qubits, (ii) the situation where one state is represented by an orthogonally prepared qubit pair, (iii) we will discuss how the problem can be generalised to qudits. In Section 3, we will discuss why homodyne measurements can be thought of as relative state measurements. We conclude in Section 4.

2 Relative state measurements

2.1 Each state is represented by identically prepared qubits

Consider the problem of estimating the angle between two directions [8]. The first direction is represented by N qubits identically prepared in some state $|\psi_1\rangle$ and the second direction is represented by M qubits identically prepared in some state $|\psi_2\rangle$. We will assume $M \geq N$. Our aim will be to estimate at best the value of $|\langle\psi_1|\psi_2\rangle|^2$. This problem can be thought of as the estimation of the state of a quantum system, i.e. the first one, relative to an axis which is itself quantum, the second system.

We will construct a positive operator valued measure $\{P(x)\}$:

$$P(x) \in \mathcal{B}(\mathcal{H}^{\otimes N+M}),\ P(x) \geq 0;\ \int_0^1 dx\ P(x) = \mathbf{1}^{\otimes N+M},$$

where $\mathcal{H}$ denotes the Hilbert space of a qubit, and $\mathcal{B}(\mathcal{H}^{\otimes N+M})$ denotes the space of (bounded) operators acting on $\mathcal{H}^{\otimes N+M}$. $\mathbf{1}$ denotes the 2×2 identity matrix.

When the outcome $P(x)$ comes out, the value x is guessed. As a figure of merit, we have chosen to consider the mean variance:

$$\Delta(P(x)) = \int dx\ d\psi_1\ d\psi_2 \\ \times \mathrm{Prob}(x|\psi_1,\psi_2)\ (x - |\langle\psi_1|\psi_2\rangle|^2)^2. \quad (1)$$

$\mathrm{Prob}(x|\psi_1,\psi_2)$ denotes the conditional probability to get the outcome x when a measurement is performed on qubits prepared in the states ψ_1, ψ_2. We have chosen this quantity because it quantifies the quality of our measurement in a satisfactory way, and because it allows us to simplify the analysis as we shall see momentarily. Note that their choices, such as the fidelity [9,10], are possible. More technically, this variance can be re-expressed as:

$$\Delta(P(x)) = \int dx\, dg_1\, dg_2\ \langle\psi_0^{\otimes N+M}|\pi_N^+(g_1)^* \otimes \pi_M^+(g_2)^* P(x) \\ \pi_N^+(g_1) \otimes \pi_M^+(g_2)|\psi_0^{\otimes N+M}\rangle (x - |\langle\psi_0|\pi(g_1)^*\pi(g_2)|\psi_0\rangle|^2)^2. \quad (2)$$

In this expression, g_1 and g_2 represent SU(2) elements and dg_1, dg_2 represent the Haar measure over $SU(2)$. π denotes the natural representation and π_N^+ denotes the irreducible representation obtained by restriction of $\pi^{\otimes N}$ onto the symmetric subspace of the space of N qubits, $\mathcal{H}_N^+$. Actually π_N^+ is the spin-j irreducible representation, with $N = 2j$. $|\psi_0\rangle$ is some fiducial state.

From any povm $P(x)$, one can construct another povm:

$$Q(x) = \int dg\ (\pi^{\otimes N}(g)^* \otimes \pi^{\otimes M}(g)^*) P(x) \\ \times (\pi^{\otimes N}(g) \otimes \pi^{\otimes M}(g)) \quad (3)$$

that achieves the same error variance as $P(x)$. Clearly, $[Q(x), \pi_N^+(g) \otimes \pi_M^+(g)] = 0 \quad \forall x, g$. Using the SU(2) Clebsch-Gordan series, the latter commutation relation can be rewritten as

$$[Q(x), \bigoplus_{k=M-N}^{M+N} \pi_k^+(g)] = 0 \qquad \forall x, g. \quad (4)$$

A nice property of the second argument of the commutator (4) is that no representation appears more than once. Consequently (Shur's lemma), $Q(x)$ has the following *diagonal* form

$$Q(x) = \sum_{k=M-N}^{M+N} q_k(x)\mathbf{1}_k, \quad (5)$$

where $\mathbf{1}_k$ is the projector onto the irreducible subspace supporting π_k^+.

The condition that $\{Q(x)\}$ should be a povm is then expressed as

$$q_k(x) \geq 0 \qquad \forall x,\ \forall k = M-N, \ldots, M+N, \quad (6)$$
$$\int_0^1 dx\, q_k(x) = 1, \qquad \forall k = M-N, \ldots, M+N. \quad (7)$$

The score $\Delta(\{Q(x)\})$ can now be written as

$$\Delta(\{Q(x)\}) = \sum_{k=M-N}^{M+N} \int_0^1 dx\ q_k(x) p_k(x), \quad (8)$$

where $p_k(x)$ is a degree-2 polynomial in x: $p_k(x) = I_k^0 x^2 - 2I_k^1 x + I_k^2$. Explicit expressions for the quantities I_k^α are given in Appendix A.

Clearly, the optimal povm is given by $q_k(x) = \delta(x - x_k^{\min})$, where $x_k^{\min}$ satisfies $p_k(x_k^{\min}) = \min_{0\leq x\leq 1} p_k(x)$.

Table 1. Minimal variance $\Delta(N,M)$.

N	1	1	1	2	2	7	20	1
M	1	2	300	2	3	7	20	∞
$\Delta^{\mathrm{opt}}(N,M)\times 10^2$	7.41	6.94	5.57	6.25	5.83	3.29	1.45	5.56

Since the p_k's are polynomials of degree 2, the values $x_k^{\min}$ are readily calculated and one finds that

$$\Delta^{\mathrm{opt}}(N,M)=\sum_{k=M-N}^{M+N}(I_k^2-\frac{(I_k^1)^2}{I_k^0}). \qquad (9)$$

Values of $\Delta^{\mathrm{opt}}(N,M)$ for some values of N,M are given in Table 1.

Let us comment a bit on equation (5). This equation tells us that the best strategy is to use a measurement whose elements are projectors onto subspaces invariant under $\pi^{\otimes N+M}$. Clearly, the space of the $N+M$ particles at hand also supports a representation of the permutation group of $N+M$ objects, $\mathrm{Sym}(N+M)$. Now it is an important result of representation theory that the algebra linearly generated by all unitaries $\pi^{\otimes N+M}(g)$ and the algebra generated by permutation operators on $\mathcal{H}^{\otimes N+M}$ are commutant of each other. Consequently, they have *common* invariant subspaces [11]. In our case, this means that the elements of our povm project onto subspaces that are invariant under permutation of particles. We interpret this fact as follows. No preferred reference frame is available to estimate the angle between two directions, but it is known which particle belongs to the set indicating the first (resp. the second) direction. Therefore, it seems natural that the only kind of properties that can be measured are those related the permutations that can be carried on the particles.

Our point is more easily illustrated in the case where $N=1, M=1$. The optimal povm is then made of two pieces, the singlet, a state which changes sign when a permutation is applied, and the triplet, which remains unchanged when a permutation is applied. Thus, our measurement actually tests permutation properties of our system, on the basis of which a guess of the relative angle is made: the singlet representation of Sym(2) makes us guess that $|\langle\psi_1|\psi_2\rangle|^2=1/3$ (the states are rather antiparallel), and the triplet representation makes us guess that $|\langle\psi_1|\psi_2\rangle|^2=5/9$ (the states are rather parallel).

2.2 One state is represented by one qubit, the other by two orthogonally prepared qubits

We now turn to the situation where one direction is specified by two anti-parallel qubits. Thus, let $\{\psi_0,\psi_1\}$ denote an orthonormal basis of $\mathcal{H}$, the Hilbert space of one qubit. One direction is specified by an element of SU(2), g_1 say, and the other direction is specified by $g_2\in \mathrm{SU}(2)$. We are now given the state $\pi(g_1)^{\otimes 2}|\psi_0,\psi_1\rangle\pi(g_2)|\psi_0\rangle$, and we want (again) to estimate at best $\langle\psi_0|\pi(g_1)^*\pi(g_2)|\psi_0\rangle$. Again, we are looking for a povm $\{P(x)\in\mathcal{B}(\mathcal{H}), P(x)\geq\ 0, \int_0^1 dxP(x)=\mathbf{1}^{\otimes 3}\}$. The figure of merit has a form similar to the one we had before:

$$\Delta=\int_0^1 dx\int dg_1\int dg_2\langle\psi_0,\psi_1,\psi_0|\pi(g_1)^{\otimes 2}\otimes\pi(g_2)P(x)$$
$$\pi(g_1)^{\otimes 2^*}\otimes\pi(g_2^*)|\psi_0,\psi_1,\psi_0\rangle(x-|\langle\psi_0|\pi(g_1)^*\pi(g_2)|\psi_0\rangle|^2)^2. \qquad (10)$$

Again we can restrict to covariant measurement and assume that

$$[P(x),\pi^{\otimes 3}(g)]=0, \qquad \forall x,g. \qquad (11)$$

The details of the extremisation can be found in Appendix D. The main result is that, perhaps surprisingly, we find essentially the same mean variance as in the case where each state is represented by two identically prepared qubits[2]. Unfortunately, we don't have any intuition on why parallel and antiparallel pairs should perform as well or not for our problem. More generally, the differences between antiparallel qubit pairs and parallel qubit pairs in quantum estimation theory are still poorly understood [12].

2.3 Generalisation to qudits

The foregoing analysis can be straightforwardly extended to qudit systems of arbitrary finite dimension d. In Appendix B, we have computed the (generalisation of the) formula (9) in the case that $N=M=1$. We have found

$$\Delta=\frac{d^2+d-2}{d(d+1)^3}. \qquad (12)$$

We see that the mean variance decreases with d as $\approx 1/d^2$. The fact that this variance should decrease with d could be expected because when the dimension increases, the overlap between two randomly drawn states tends (on average) to 0, i.e. the states are increasingly orthogonal, and thus easier to estimate. We also note that the povm consists again on projectors onto subspaces invariant under permutations of particles, i.e. the overlap between two quantum states is estimated upon testing permutation properties.

3 Homodyne detection

We now wish to describe how homodyne detection can be thought of as a relative state measurement (see also Refs. [13,14]). In a homodyne measurement [15], two e-m fields impinge on the two input ports of a balanced beam

[2] This result is consistent with what has been found in [8,9]. Actually, careful calculations indicate tiny differences between the anti-parallel case and the parallel case. The antiparallel case exhibits a slightly lower variance (a difference emerges from the tenth digit). In contrast, if one considers the fidelity as a figure of merit as in [10], then parallel pairs are slightly better. We have not investigated these differences further.

splitter. One of the input is generally referred to as "signal", and the other as "reference". We will assume that the signal and the reference have the same frequency and the same polarisation. A photodetector is placed at each output port of the beam splitter. This scheme aims at measuring a quadrature of the signal field from the difference of the two photocurrents read on the detectors. Let a, b denote the annihilation operators for the two input ports, and c, d the annihilation operators for the output ports. The observable that is actually measured by the homodyne set-up is

$$c^*c - d^*d = a^*b + b^*a, \tag{13}$$

where $c = (a+b)/\sqrt{2}$, $d = (a-b)/\sqrt{2}$.

The reference field is assumed to be in a coherent state $|\psi_r\rangle = |\beta e^{i\theta}\rangle$, where β and θ are two known real numbers. It is also assumed that β is so large that $b \approx \langle b\rangle = \beta e^{i\theta}$, i.e. the reference is a *classical* field. Then the homodyne set-up measures the observable $\beta(a^* e^{i\theta} + a e^{-i\theta})$, and thus indeed corresponds to measuring a quadrature of the signal field. We can choose the quadrature we wish to measure upon tuning the phase θ.

Let $|\psi_s\rangle = \sum_n \psi_n (a^{*n}/\sqrt{n!})|\mathrm{vac}\rangle$ denote the state of the signal. It is a remarkable fact that the probability to get a given outcome, say K, is invariant under the transformation $a \to e^{i\phi}a, b \to e^{i\phi}b$, for arbitrary values of ϕ, or equivalently

$$|\psi_r\rangle = |\beta e^{i\theta}\rangle \to |\psi_r(\phi)\rangle = |\beta e^{i(\theta+\phi)}\rangle,$$
$$|\psi_s\rangle = \sum_n \psi_n \frac{a^{*n}}{\sqrt{n!}}|\mathrm{vac}\rangle \to |\psi_s(\phi)\rangle = \sum_n \psi_n \frac{a^{*n}e^{in\phi}}{\sqrt{n!}}|\mathrm{vac}\rangle.$$

Thus, whatever convention we choose for the absolute phase of the e-m field, this convention does *not* affect in any manner the consistency of homodyne measurement. For example, we could choose the convention where the field is described by

$$\int_0^{2\pi} \frac{d\phi}{2\pi} |\psi_r(\phi)\rangle\langle\psi_r(\phi)| \otimes |\psi_s(\phi)\rangle\langle\psi_s(\phi)|. \tag{14}$$

We will restrict the remaining of the discussion to the case where the state we want to measure is a coherent state that we will denote $|\alpha\rangle$. Then, one can re-write the state (14) in number state basis as

$$\sum_{k=0}^{\infty} \frac{e^{-(|\alpha|^2+|\beta|^2)}\sqrt{|\alpha|^2+|\beta|^2}}{k!}|k\rangle_c\langle k|, \tag{15}$$

where

$$|k\rangle_c = \frac{(\alpha a^* + \beta b^*)^k}{\sqrt{|\alpha|^2+|\beta|^2}\sqrt{k!}}|\mathrm{vac}\rangle$$

denotes a state of k photons in the mode $(\alpha a^* + \beta b^*)/\sqrt{|\alpha|^2+|\beta|^2}$.

Assume that the mean photon number $|\alpha|$ is known, $\arg(\alpha)$ is the quantity (phase) we wish to measure. But with respect to what? To a reference $|\beta\rangle$. In words, instead of thinking of a signal and a reference system, we can equivalently think of a Poisson distribution of qubits all in the state $|\psi\rangle \propto \alpha|0\rangle + \beta|1\rangle$. In this description, the difference between the reference and the signal has completely disappeared. If the mean number of photons $|\beta|^2$ is known and very large, then homodyne measurement turns to be an estimation problem for qubits on a circle of the Bloch sphere [12].

4 Conclusion

In summary, we have considered relative state estimation problems, where the reference system is itself quantum. We emphasized how general this concept of relative state is and that it conveys an aspect of entanglement dual to the most studied quantum correlation between subsystems. More specifically, we have investigated the problem of estimating the overlap between two (pure) quantum states in various scenarii. In the case where each state is represented by identically prepared qubits, we have noticed a connection between optimal strategies and measurements testing permutation properties of the systems at hand. It would be interesting to investigate this connection further in other estimation problem. We have also seen antiparallel qubit pairs and parallel qubit pairs play are equivalent when used as a reference axis with respect to which a qubit is measured. It is an interesting open problem to provide a qualitative explanation for this fact. We have also revisited homodyne measurements, and discuss why it is a relative state measurement.

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Appendix A: Evaluations of $\mathbf{I_k^i}$

Let us start with I_k^0. From Schur's lemma, we find that

$$I_k^0 = \frac{1}{\dim\mathcal{H}_M^+} \operatorname{tr}(\mathbf{1}_k(|\psi_0^{\otimes N}\rangle\langle\psi_0^{\otimes N}| \otimes \mathbf{1}_M)). \tag{A.1}$$
$$= \frac{1}{\dim\mathcal{H}_M^+} \int_{SU(2)} dg \operatorname{tr}(\mathbf{1}_k(\pi_N^+(g)|\psi_0\rangle\langle\psi_0|^{\otimes N} \times \pi_N^+(g)^* \otimes \mathbf{1}_M)) \tag{A.2}$$
$$= \frac{\operatorname{tr}\mathbf{1}_k}{\dim\mathcal{H}_M^+ \dim\mathcal{H}_N^+}. \tag{A.3}$$

Similarly, one shows that

$$I_k^1 = \frac{1}{\dim\mathcal{H}_{M+1}^+} \operatorname{tr}((\mathbf{1}_k \otimes |\psi_0\rangle\langle\psi_0|) \times (|\psi_0^{\otimes N}\rangle\langle\psi_0^{\otimes N}| \otimes \mathbf{1}_{M+1})), \tag{A.4}$$

$$I_k^2 = \frac{1}{\dim\mathcal{H}_{M+2}^+}\,\mathrm{tr}((\mathbf{1}_k \otimes |\psi_0^{\otimes 2}\rangle\langle\psi_0^{\otimes 2}|) \times (|\psi_0^{\otimes N}\rangle\langle\psi_0^{\otimes N}| \otimes \mathbf{1}_{M+2})). \quad (A.5)$$

Straightforwardly, $I_k^0 = (k+1)/[(N+1)(M+1)]$. Unfortunately, we were not able to find expressions as simple for I_j^1 and I_j^2. However, a direct computation shows that

$$I_j^1 = \frac{1}{M+2}\sum_{m=-j}^{+j} |C^{(j,m)}_{(N/2,N/2)(M/2,m-N/2)}|^2 \times \left|C^{((M+1)/2,m-N/2+1/2)}_{(M/2,m-N/2)(1/2,1/2)}\right|^2, \quad (A.6)$$

$$I_j^2 = \frac{1}{M+3}\sum_{m=-j}^{+j} |C^{(j,m)}_{(N/2,N/2)(M/2,m-N/2)}|^2 \times \left|C^{((M+2)/2,m-N/2+1)}_{(M/2,m-N/2)(1,1)}\right|^2, \quad (A.7)$$

where $C^{j,m}_{(j_1,m_1)(j_2,m_2)} = \langle j,m|j_1,m_1;j_2,m_2\rangle$ denote Clebsch-Gordan coefficients.

Appendix B: Generalisation to qudits

The irreducible representations of SU(d) are labelled by d-uples of positive integers $m_1,\ldots,m_d$ satisfying $m_1 \geq \ldots \geq m_d$ [11]. These d-uples are called the highest weights of the representations. Moreover, we can always choose $m_d = 0$. The Clebsch-Gordan series for $\pi_N^+ \otimes \pi_M^+$ now reads

$$\pi_N^+\otimes\pi_M^+ \approx \bigoplus_{0\leq k\leq \min\{M,N\}} \pi(M+N-k,k,0,\ldots,0). \quad (B.1)$$

Again, no representation appears more than once in this series, so that $Q(x)$ assumes again a diagonal form.

The relations (A.4, A.5) still hold. But giving the analogue of equations (A.6, A.7) involves dealing with SU(d) Clebsch-Gordan coefficients for $d > 2$, which is a heavy business. Therefore we didn't carry our analysis as far as for the qubit case. There are however some interesting situations where the expressions (A.4, A.5) can be calculated relatively easily, such as the case where $N = M = 1$, which we will discuss now. First we need expressions for $\mathbf{1}(1,1,0,\ldots,0)$, the projector onto the antisymmetric subspace of two qudits, $\mathbf{1}(2,0,0,\ldots,0)$, the projector onto the symmetric subspace of two qudits, and $\mathbf{1}(3,0,0,\ldots,0)$, the projector onto the symmetric subspace of three qudits. We have:

$$\mathbf{1}(1,1,0,\ldots,0) = \frac{1}{2}\sum_{k,l=1}^{d}(|kl\rangle - |lk\rangle)\langle kl|, \quad (B.2)$$

$$\mathbf{1}(2,0,\ldots,0) = \frac{1}{2}\sum_{k,l=1}^{d}(|kl\rangle + |lk\rangle)\langle kl|, \quad (B.3)$$

$$\mathbf{1}(3,0,\ldots,0) = \frac{1}{6}\sum_{k,l,m=1}^{d}(|klm\rangle + |kml\rangle + |lmk\rangle + |lkm\rangle + |mkl\rangle + |mlk\rangle)\langle klm|. \quad (B.4)$$

Then, using the fact that $\dim\mathcal{H}_N^+ = (d+N-1)!/N!(d-1)!$ [11], we find

$$I^0(1,1,0,\ldots,0) = \frac{1}{(\dim\mathcal{H})^2}\,\mathrm{tr}\,\mathbf{1}(1,1,0,\ldots,0) = \frac{d-1}{2d}, \quad (B.5)$$

$$I^0(2,0,\ldots,0) = \frac{1}{(\dim\mathcal{H})^2}\,\mathrm{tr}\,\mathbf{1}(2,0,0,\ldots,0) = \frac{d+1}{2d}, \quad (B.6)$$

$$I^1(1,1,0,\ldots,0) = \frac{1}{\dim\mathcal{H}_2^+}\,\mathrm{tr}((\mathbf{1}(1,1,0,\ldots,0)\otimes|\psi_0\rangle\langle\psi_0|) \times (|\psi_0\rangle\langle\psi_0| \otimes \mathbf{1}(2,0,\ldots,0))) = \frac{d-1}{2d(d+1)}, \quad (B.7)$$

$$I^1(2,0,\ldots,0) = \frac{1}{\dim\mathcal{H}_2^+}\,\mathrm{tr}((\mathbf{1}(2,0,\ldots,0)\otimes|\psi_0\rangle\langle\psi_0|) \times (|\psi_0\rangle\langle\psi_0| \otimes \mathbf{1}(2,0,\ldots,0))) = \frac{d+3}{2d(d+1)}, \quad (B.8)$$

$$I^2(1,1,0,\ldots,0) = \frac{1}{\dim\mathcal{H}_3^+}\,\mathrm{tr}((\mathbf{1}(1,1,0,\ldots,0) \otimes |\psi_0\rangle\langle\psi_0|^{\otimes 2})(|\psi_0\rangle\langle\psi_0|\otimes\mathbf{1}(3,0,\ldots,0))) = \frac{d-1}{d^3+3d^2+2d}, \quad (B.9)$$

$$I^2(2,0,\ldots,0) = \frac{1}{\dim\mathcal{H}_3^+}\,\mathrm{tr}((\mathbf{1}(2,0,\ldots,0)\otimes|\psi_0\rangle\langle\psi_0|^{\otimes 2}) \times (|\psi_0\rangle\langle\psi_0| \otimes \mathbf{1}(3,0,\ldots,0))) = \frac{d+5}{d^3+3d^2+2d}. \quad (B.10)$$

From these identities, we can compute the formula (9) and obtain the mean variance (12).

Appendix C: The asymptotic limit

Suppose that one direction is specified by one qubit, and the other by an infinite number of identically prepared qubits. We can thus suppose that this second direction, that we choose to call z, is known with arbitrary precision [16]. We can therefore imagine that the first step of our measurement consists in building a *classical* system that will serve as a z-axis. We are thus (again) looking for a povm $\{P(x)\}$ satisfying the conditions

$$0 \leq P(x) \leq \mathbf{1}, \qquad \int_0^1 dx P(x) = \mathbf{1}. \quad (C.1)$$

As a figure of merit, we will consider

$$\int dg \int_0^1 dx \langle\psi_0|\pi(g)^* P(x)\pi(g)|\psi_0\rangle(x - |\langle\psi_0|\pi(g)|\psi_0\rangle|^2)^2. \quad (C.2)$$

For any povm $\{P(x)\}$, the povm whose elements are

$$Q(x) = \int \frac{d\theta}{2\pi} e^{-i\theta\sigma_z} P(x) e^{i\theta\sigma_z}$$

achieves the same score. We can thus assume that $P(x)$ is diagonal in the z-basis:

$$\begin{pmatrix} s_0(x) & 0 \\ 0 & s_1(x) \end{pmatrix}.$$

The mean error can again be written as

$$\Delta = \int_0^1 dx(I_0(x)x^2 - 2I_1(x)x + I_2(x)). \quad \text{(C.3)}$$

Let us calculate $I_0(x), I_1(x), I_2(x)$. One readily checks that

$$I_0(x) = \operatorname{tr} P(x) \int d g \pi(g)|\psi_0\rangle\langle\psi_1|\pi(g)^*, \quad \text{(C.4)}$$
$$I_1(x) = \operatorname{tr}[(P(x) \otimes |\psi_0\rangle\langle\psi_0|) \times \int dg(\pi^{\otimes 2}(g)|\psi_0^{\otimes 2}\rangle\langle\psi_0^{\otimes 2}|\pi^{\otimes 2}(g)^*)], \quad \text{(C.5)}$$
$$I_2(x) = \operatorname{tr}[(P(x) \otimes |\psi_0^{\otimes 2}\rangle\langle\psi_0^{\otimes 2}|) \times \int dg(\pi^{\otimes 3}(g)|\psi_0^{\otimes 3}\rangle\langle\psi_0^{\otimes 3}|\pi^{\otimes 3}(g)^*)]. \quad \text{(C.6)}$$

Using Shur's lemma, we get

$$I_0(x) = \frac{1}{2}\operatorname{tr} P(x) = \frac{1}{2}(s_0(x) + s_1(x)), \quad \text{(C.7)}$$
$$I_1(x) = \frac{1}{3}\operatorname{tr}[(P(x) \otimes |\psi_0\rangle\langle\psi_0|)S_2] = \frac{1}{3}(s_0(x) + \frac{1}{2}s_1(x)), \quad \text{(C.8)}$$
$$I_2(x) = \frac{1}{4}\operatorname{tr}[(P(x) \otimes |\psi_0^{\otimes 2}\rangle\langle\psi_0^{\otimes 2}|)S_3] = \frac{1}{4}(s_0(x) + \frac{1}{3}s_1(x)). \quad \text{(C.9)}$$

We can then simply write

$$\Delta = \int_0^1 dx\left[s_0(x)\left(\frac{x^2}{2} - \frac{2}{3}x + \frac{1}{4}\right) + s_1(x)\left(\frac{x^2}{2} - \frac{x}{3} + \frac{1}{12}\right)\right], \quad \text{(C.10)}$$

from which we infer that the optimal povm is given by $s_0(x) = \delta(x-2/3), s_1(x) = \delta(x-1/3)$. In turn the optimal variance is $\Delta = 1/18 \approx .0555$.

Appendix D: The antiparallel case

The Clebsch-Gordan series for $\pi^{\otimes 3}$ reads $\pi_1^+ \oplus \pi_1^+ \oplus \pi_3^+$. The problem is now more complicated because the representation π_1^+ appears more than once. As a result, the povm elements do not have an a priori diagonal form anymore, but only a block-diagonal form (in the basis corresponding to the irreducible representations):

$$Q(x) = \begin{pmatrix} q_{00}(x)\mathbf{1}_{00} & q_{01}(x)\mathbf{1}_{01} & \mathbf{0} \\ q_{10}(x)\mathbf{1}_{10} & q_{11}(x)\mathbf{1}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & q_{33}(x)\mathbf{1}_{33} \end{pmatrix}. \quad \text{(D.1)}$$

Explicit expressions for the operators $\mathbf{1}_{00}, \mathbf{1}_{01}, \mathbf{1}_{10}, \mathbf{1}_{11}, \mathbf{1}_{33}$ will be given below.

There exist again functions $I_0(x), I_1(x), I_2(x)$ such that we can write the average error as $\Delta = \int_0^1 (I_2(x) - 2I_1(x)x + I_0(x)x^2)$. Now,

$$I_i = \int dg\langle\psi_0, \psi_1, \psi_0|(\mathbf{1}^{\otimes 2} \otimes \pi(g)^*)Q(x)(\mathbf{1}^{\otimes 2} \otimes \pi(g)) \times |\psi_0, \psi_1, \psi_0\rangle|\langle\psi_0\pi(g)|\psi_0\rangle|^{2i}, \quad \text{(D.2)}$$

where $i = 0, 1, 2$. We compute these expressions explicitly. In the following, $\{|0\rangle, |1\rangle\}$ will denote an orthonormal basis of the Hilbert space of one qubit. We start with $I_0(x)$.

$$I_0(x) = \operatorname{tr} Q(x)(|01\rangle\langle 01| \otimes \frac{\mathbf{1}}{2}) = \operatorname{tr} Q(x)\left(\int dg\pi^{\otimes 2}(g)|01\rangle\langle 01|\pi^{\otimes 2}(g)^* \otimes \frac{\mathbf{1}}{2}\right). \quad \text{(D.3)}$$

Due to Shur's lemma, there exists constants γ_0 and γ_2 such that $\int dg\pi^{\otimes 2}(g)|01\rangle\langle 01|\pi^{\otimes 2}(g)^* = \gamma_0 S_0 + \gamma_2 S_2$. S_0 (resp. S_2) is the projector onto the antisymmetric (resp. symmetric) subspace of two-qubits. Their component (in a computational basis) are given in terms of Clebsch Gordan-coefficients as

$$\langle uv|S_2|rs\rangle \equiv T_{rs}^{uv} = C_{(1r)(1s)}^{(2j)} C_{(2j)}^{(1u)(1v)},$$
$$\langle uv|S_0|rs\rangle \equiv A_{rs}^{uv} = C_{(1r)(1s)}^{(00)} C_{(00)}^{(1u)(1v)}.$$

(Sum over repeated indices is understood.) The constants γ_0 and γ_2 are then easily calculated: $\gamma_0 = \langle 01|S_0|01\rangle/\operatorname{tr} S_0 = 1/2$, $\gamma_2 = \langle 01|S_2|01\rangle/\operatorname{tr} S_2 = 1/6$. Defining $Q_{rst}^{uvw}(x) = \langle uvw|Q(x)|rst\rangle$, one finds that

$$I_0(x) = \frac{1}{4}Q_{rsw}^{uvw}(x)(\frac{1}{3}T_{uv}^{rs} + A_{uv}^{rs}). \quad \text{(D.4)}$$

Similarly, one finds that

$$I_1(x) = \frac{1}{3}\operatorname{tr}(Q(x) \otimes |0\rangle\langle 0|)(|01\rangle\langle 01| \otimes S_2) = \frac{1}{3}Q_{01t}^{01w}(x)T_{w0}^{t0}, \quad \text{(D.5)}$$

and

$$I_2(x) = \frac{1}{3}\operatorname{tr}(Q(x) \otimes |00\rangle\langle 00|)(|01\rangle\langle 01| \otimes S_3) = \frac{1}{4}Q_{01t}^{01w}(x)B_{w0}^{t0}, \quad \text{(D.6)}$$

where $B_{rs}^{uv} = C_{(1r)(2s)}^{(3j)} C_{(3j)}^{(1u)(2v)}$.

In the computational basis, the operators $\mathbf{1}_{ij}$ are explicitly given by

$$(\mathbf{1}_{00})_{xyz}^{uvw} = C_{(1x)(1y)}^{(00)} C_{(00)(1z)}^{(1j)} C_{(00)}^{(1u)(1/2v)} C_{(1j)}^{(00)(1w)}, \quad \text{(D.7)}$$
$$(\mathbf{1}_{11})_{xyz}^{uvw} = C_{(1x)(1y)}^{(2m)} C_{(2m)(1z)}^{(1j)} C_{(2k)}^{(1u)(1v)} C_{(1j)}^{(2k)(1w)}, \quad \text{(D.8)}$$
$$(\mathbf{1}_{01})_{xyz}^{uvw} = C_{(1x)(1y)}^{(2m)} C_{(2m)(1z)}^{(1j)} C_{(00)}^{(1u)(1v)} C_{(1j)}^{(00)(1w)}, \quad \text{(D.9)}$$
$$(\mathbf{1}_{10})_{xyz}^{uvw} = [(\mathbf{1}_{01})_{xyz}^{uvw}]^*, \quad \text{(D.10)}$$
$$(\mathbf{1}_{33})_{xyz}^{uvw} = C_{(1x)(1y)}^{(1m)} C_{(1m)(1z)}^{(3j)} C_{(2k)}^{(1u)(1v)} C_{(3j)}^{(2k)(1w)}. \quad \text{(D.11)}$$

With all the information that we have gathered, an explicit calculation can now be carried to find

$$I_0(x) = \frac{1}{2}q_{00}(x) + \frac{1}{6}q_{11}(x) + \frac{1}{3}q_{33}(x), \tag{D.12}$$

$$I_1(x) = \frac{1}{4}q_{00}(x) + \frac{1}{12}q_{11}(x) - \frac{1}{12\sqrt{3}}(q_{01}(x) + q_{10}(x)) + \frac{1}{6}q_{33}(x), \tag{D.13}$$

$$I_2(x) = \frac{1}{6}q_{00}(x) + \frac{1}{18}q_{11}(x) - \frac{1}{12\sqrt{3}}(q_{01}(x) + q_{10}(x)) + \frac{1}{9}q_{33(x)}. \tag{D.14}$$

As is obvious from the block-diagonal form of $Q(x)$, the error Δ can be decomposed as $\Delta = \Delta_1 + \Delta_3$. $\Delta_3 = \int_0^1 dx q_{33}(x)(1/9 - h/3 + h^2/3)$ and Δ_1 can be conveniently written as $\Delta_1 = \int_0^1 dx \,\mathrm{tr}\, \tilde{Q}(x)F(x)$, where

$$\tilde{Q}(x) = \begin{pmatrix} q_{00}(x) & q_{01}(x) \\ q_{10}(x) & q_{11}(x) \end{pmatrix},$$

and where

$$f_{00}(x) = \frac{1}{6} - \frac{1}{2}x + \frac{1}{2}x^2,$$

$$f_{11}(x) = \frac{1}{18} - \frac{1}{6}x + \frac{1}{6}x^2,$$

$$f_{01}(x) = f_{10}(x) = -\frac{1}{6\sqrt{3}}x.$$

Δ_{33} can be readily extremised, setting $q_{33}(x) = \delta(x - 1/2)$, giving $\Delta_{33} = 1/36$.

The extremisation of Δ_1 is less straightforward. If we restrict to povm's with a finite number of outcomes, then the solutions are of the form

$$\tilde{Q}^i(x) = w_i^2 \delta(x - x_i)\frac{1}{2}(\mathbf{1} + n_1^i X_1 + n_2^i X_2 + x_3^i X_3), \quad i = 1 \ldots I. \tag{D.15}$$

where $\int_0^1 dx \sum_{i=1}^{\nu} \tilde{Q}^i(x) = \mathbf{1}$, and where X_1, X_2, X_3 denote the three Pauli matrices.

Then minimising Δ_{01} amounts to minimise

$$\frac{1}{2}\sum_{i=1}^{\nu} w_i^2 f_{00}(x_i) + f_{11}(x_i) + n_1^i(f_{10}(x_i) + f_{01}(x_i)) + n_3^i(f_{00}(x_i) - f_{11}(x_i)), \tag{D.16}$$

with the constraints $\sum_{i=1}^{\nu} w_i^2 = 2$, $\sum_{i=1}^{\nu} w_i^2 n_j^i = 0$, $\forall j = 1,2,3$. For $\nu = 2$, our numerical extremisation has yielded the following povm:

$$\tilde{Q}^1(x) = \frac{1}{2}\delta(x - x_1)(\mathbf{1} + X_1), \tag{D.17}$$

$$\tilde{Q}^2(x) = \frac{1}{2}\delta(x - x_2)(\mathbf{1} - X_1), \tag{D.18}$$

where $x_1 = 0.644338...$ and $x_2 = 0.355662...$ Very interestingly, the total optimal score $\Delta_{01} + \Delta_{33}$ equals $\Delta^{\text{parallel}}(2,1)$. We also wondered whether increasing the number of outcomes for the part Δ_{01} could decrease the overall score. Looking for povms with more outcome, we have found no improvement. We therefore believe that the povm (D.17) is indeed optimal.

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